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The ε-Entropy of Some Compact Subsets of l^p

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A block $B\{a_k\} = \{\{x_k\} : |x_k| \leqslant a_k, \{a_k\} \in l^p, a_k \downarrow\}$ is a compact subset of l^p. The ellipsoid $E\{b_k\}$ is the compact subset of l^p given by $\{\{x_k\} \in l^p : \Sigma |x_k/b_k|^p \leqslant 1, b_k \downarrow 0\}$. Let $H(S, \epsilon)$ be the ϵ-entropy of the set S with respect to the l^p metric. The following results are obtained:

$$\int_0 H^{1/q}(B\{a_k\}, \epsilon)\, d\epsilon < \infty \Leftrightarrow \Sigma a_k < \infty,$$

$$\int_0 H(E\{b_k\}, \epsilon)\, d\epsilon^p < \infty \Leftrightarrow \Sigma b_k^p < \infty,$$

$1 < p < \infty, q = p/(p-1)$.

1. RESULTS AND DISCUSSION

Let $\{a_k\}$ be a nonincreasing sequence, $\{a_k\} \in l^p$, $1 < p < \infty$. We shall consider two compact subsets of l^p: a block $B\{a_k\}$ and an ellipsoid $E\{b_k\}$, defined as follows:

$$B\{a_k\} = \{\{x_k\}: |x_k| \leqslant a_k\},$$

$$E\{b_k\} = \left\{\{x_k\} \in l^p: \sum_{k=1}^{\infty} \left|\frac{x_k}{b_k}\right|^p \leqslant 1, b_k \downarrow 0\right\}.$$

Let S be a compact subset of l^p and $N(S, \epsilon)$ the smallest number of sets of diameter at most 2ϵ that cover S. The metric is the l^p metric, i.e., the covering sets are translations of the sets $\{\{\xi_k\} \in l^p: (\sum |\xi_k|^p)^{1/p} \leqslant \epsilon\}$. The ϵ-entropy of S, $H(S, \epsilon)$, is defined as $\log N(S, \epsilon)$.

Motivated by a problem in probability to which we shall refer below, we obtain necessary and sufficient conditions on the ϵ-entropy of $B\{a_k\}$ which imply that $\sum a_k < \infty$, and necessary and sufficient conditions on the ϵ-entropy of $E\{b_k\}$ which imply that $\sum b_k^p < \infty$. Our results are:

THEOREM 1. *Let $\{a_k\}$ be nonincreasing, $\{a_k\} \in l^p$, $1 < p < \infty$, and let $H(B\{a_k\}, \epsilon)$ be the ϵ-entropy of $B\{a_k\}$ with respect to the l^p metric; then*

$$\int_0 H^{1/q}(B\{a_k\}, \epsilon)\, d\epsilon < \infty \Leftrightarrow \sum a_k < \infty,$$

where $q = p/(p-1)$.

THEOREM 2. *Let $E\{b_k\}$ be an l^p ellipsoid and $H(E\{b_k\})$ its ϵ-entropy with respect to the l^p metric; then*

$$\int_0 H(E\{b_k\}, \epsilon)\, d\epsilon^p < \infty \Leftrightarrow \sum b_k{}^p < \infty.$$

From Theorem 1, *we shall deduce the corollary.*

COROLLARY. *Let $\{a_k\}$ be a nonincreasing sequence, $\{a_k\} \in l^p$, $1 < p < \infty$. Let*

$$d_n = \left(\sum_{k=n+1}^{\infty} a_k{}^p\right)^{1/p} \qquad \text{and} \qquad M(y) = \sup\{n\colon d_n \geqslant y\};$$

then

$$\int_0 M^{1/q}(y)\, dy < \infty \Leftrightarrow \int_0 H^{1/q}(B\{a_k\}, \epsilon)\, d\epsilon < \infty,$$

where $q = p/(p-1)$.

In [1] Mitjagin gives estimates for the ϵ-entropy of various subsets of l^p. Theorems 1 and 2 are not so much estimates of the ϵ-entropy of blocks and ellipsoids as characterizations of certain geometric properties of $B\{a_k\}$ and $E\{b_k\}$ by conditions on their ϵ-entropy. Lorentz [9] estimates the ϵ-entropy of full approximation sets in arbitrary Banach spaces. His results are independent of the norm of the Banach space while Theorem 1 depends on p.

A Gaussian process can be described as a linear map from a Hilbert space H into real-valued Gaussian random variables. A block $B\{a_k\}$ in H will map into a continuous Gaussian process iff $\sum a_k < \infty$, and an ellipsoid $E\{b_k\}$ will map into a continuous Gaussian process iff $\sum b_k{}^2 < \infty$. Therefore, Theorems 1 and 2 with $p = q = 2$ give necessary and sufficient conditions for the processes that are images of $B\{a_n\}$ and $E\{b_n\}$ to be continuous, in terms of the ϵ-entropy of B and E (see [2–5] for details). In [5] Theorem 1 is obtained in the case $p = 2$; however, the implication going to the right is proved by a probabilistic argument which doesn't seem to be extend to l^p for $p \neq 2$. Attempting to find a nonprobabilistic proof of this result led to this work.

In the case $p = 2$, Theorem 2 is stated by Sudakov [4]. Our proof is a simple application of Mitjagin's ([1], Theorem 3) estimate for the ϵ-entropy of $E(\{b_n\})$ in l^p.

The corollary is mentioned as a potentially useful curiosity. Generally, for $\sum a_k < \infty$, $M(\epsilon)$ is much larger than $H(\epsilon)$; however, when $\{a_k\}$ is fairly smooth and close to being a divergent sequence (i.e., $a_n = 1/n^\alpha$, $\alpha > 1$), M is comparable to H.

The results in this paper are obtained using techniques developed in Boas and Marcus [6] for examining the mutual convergence or divergence of integrals involving a monotone function and its generalized inverse. The author is grateful for Professor Boas' assistance in this work.

2. Proofs

The following lemmas are used in the proofs.

Lemma 1. *Let $G(x)$ be a nonincreasing real-valued function on $[a, \infty)$ and $G^{-1}(y)$ a generalized inverse of G; then the following integrals are either all convergent or all divergent*:

$$\int^{\infty} G(x)\,dx,\quad \int^{\infty} x\,dG(x),\quad \int_0 G^{-1}(y)\,dy.$$

This lemma is proved for the first and third integral in [6]. The proof is simply an application of integration by parts. The same argument applies to the second integral.

Lemma 2. *Let $\{a_k\} \in l^p$, $1 < p \leqslant \infty$, be nonincreasing. Define*

$$\delta_n = \left(\frac{1}{n}\sum_{k=n}^{\infty} a_k{}^p\right)^{1/p}, \qquad \textit{then}\quad \sum \delta_n < \infty \Leftrightarrow \sum a_n < \infty.$$

The implication to the right is Theorem 345, p. 255, in Hardy, Littlewood, and Polya [7]. The converse is due to Boas [8].

Lemma 3. *Let f be a continuous function and g a piecewise strictly monotone continuous function*;

$$g(0) = 0 = \min_{0 \leqslant x \leqslant \delta} g(x); \qquad g(\delta) = \gamma = \max_{0 \leqslant x \leqslant \delta} g(x).$$

Then

$$\int_0^{\gamma} f(u)\,du = \int_0^{\delta} f(g(u))\,dg(u).$$

Proof. A piecewise monotone function has a countable number of local maxima and minima. Suppose that $g(u)$ is decreasing in $[\delta_1, \delta_2]$; then there is another interval $[\delta_0, \delta_1]$ such that g is increasing in $[\delta_0, \delta_1]$ and $g(\delta_0) = g(\delta_2)$. Since g is monotone in each interval

$$\int_{\delta_0}^{\delta_1} f(g(u))\, dg(u) = \int_{g(\delta_0)}^{g(\delta_1)} f(s)\, ds = \int_{g(\delta_2)}^{g(\delta_1)} f(s)\, ds = -\int_{\delta_1}^{\delta_2} f(g(u))\, dg(u). \quad (2.1)$$

Since $g(\delta) = \gamma$ and $g(0) = 0$, we can began at δ and integrate backwards applying 2.1 when necessary. The result follows.

Proof of Theorem 1. Let $B\{a_k\}$ be a block in l^p with $\sum a_k < \infty$. It is convenient to eliminate large jumps in $\{a_k\}$. Construct a nonincreasing sequence $\{b_k\}$, $b_k \geqslant a_k$, such that $b_k \leqslant 2b_{k+1}$. Then

$$\left(\sum_{k=n}^{\infty} b_k^{\,p}\right)^{1/p} \leqslant 3\left(\sum_{k=n+1}^{\infty} b_k^{\,p}\right)^{1/p}. \quad (2.2)$$

The sequence $\{b_k\}$ can be constructed so that $\sum b_k < \infty$. $H(B\{b_k\}, \epsilon)$, the ϵ-entropy of $B\{b_k\}$ with respect to the l^p metric, is larger than $H(B\{a_k\}, \epsilon)$ since $B\{a_k\} \subset B\{b_k\}$. We will show that

$$\int_0^{\delta} H^{1/q}(B\{b_n\}, \epsilon)\, d\epsilon < \infty,$$

which implies $\int_0^{\delta} H^{1/q}(B\{a_n\}, \epsilon)\, d\epsilon < \infty$.

Let

$$\delta_n = \left(\frac{1}{n+1} \sum_{k=n+1}^{\infty} b_k^{\,p}\right)^{1/p} \qquad \text{and} \qquad \epsilon_n = \left(2 \sum_{k=n+1}^{\infty} b_k^{\,p}\right)^{1/p}.$$

Since the δ_n are decreasing, if $b_k/\delta_n \geqslant 1$, then $b_k/\delta_{n+1} \geqslant 1$. Let

$$M(n) = \min\left\{n, \max\left\{k: \frac{b_k}{\delta_n} \geqslant 1\right\}\right\}. \quad (2.3)$$

We shall now find an upper bound $\overline{N}(\epsilon_n)$ for $N(B\{b_k\}, \epsilon_n)$. For n fixed, consider the following elements in $B\{b_k\}$:

$$\{j_1\delta_n, \ldots, j_{M(n)}\delta_n\} \text{ the } j_k \text{ are integers} \quad (2.4)$$

$$0 \leqslant |j_k| \leqslant \left[\frac{b_k}{\delta_n}\right], \qquad k = 1, \ldots, M(n);$$

([ ] denotes integral part).

Let $x \in B\{b_k\}$. There is an element in (2.4), call it $\bar{x}$ for which

$$\| x - \bar{x} \|^p \leqslant M(n)\, \delta_n{}^p + \sum_{k=M(n)+1}^{n} b_k{}^p + \sum_{k=n+1}^{\infty} b_k{}^p,$$

where $\| \ \|$ refers to the l^p norm. By (2.3),

$$\| x - \bar{x} \|^p \leqslant n\delta_n{}^p + \sum_{k=n+1}^{\infty} b_k{}^p \leqslant 2 \sum_{k=n+1}^{\infty} b_k{}^p,$$

that is, $\| x - \bar{x} \| \leqslant \epsilon_n$.

The covering sets are of the form $\bar{x} + S$ where $\bar{x}$ is an element of (2.4) and S is the following neighborhood of the origin.

$$S = \{\{x_i\}: | x_i | \leqslant \delta_n , i = 1,..., n; | x_i | \leqslant b_i , i > n\}.$$

$N(B\{b_k\}, \epsilon_n)$ is less than the number of elements in (2.4), i.e.,

$$N(B\{b_k\}, \epsilon_n) \leqslant \prod_{k=1}^{M(n)} 2 \left(\frac{b_k}{\delta_n} + 1\right) \leqslant \bar{N}(\epsilon_n) = 4^n \prod_{k=1}^{M(n)} \frac{b_k}{\delta_n}.$$

Let $\bar{H}(\epsilon_n) = \log \bar{N}(\epsilon_n)$, for some N dependent on δ

$$\begin{aligned}\int_0^{\delta} H^{1/q}(B\{b_k\}, \epsilon)\, d\epsilon &\leqslant \sum_{n=N}^{\infty} H^{1/q}(B\{b_k\}, \epsilon_n)[\epsilon_{n-1} - \epsilon_n] \\ &\leqslant \sum_{n=N}^{\infty} \bar{H}^{1/q}(\epsilon_n)[\epsilon_{n-1} - \epsilon_n] \\ &\leqslant \epsilon_{N-1}\bar{H}^{1/q}(\epsilon_N) + \sum_{n=N}^{\infty} \epsilon_n[\bar{H}^{1/q}(\epsilon_{n+1}) - \bar{H}^{1/q}(\epsilon_n)]. \qquad (2.5)\end{aligned}$$

This section of the proof is completed by showing that $\sum b_k < \infty$ implies that the final sum in (2.5) is finite.

$$\begin{aligned}\sum_{n=N}^{\infty} \epsilon_n[\bar{H}^{1/q}(\epsilon_{n+1}) - \bar{H}^{1/q}(\epsilon_n)] &\leqslant 1/q \sum_{n=N}^{\infty} \epsilon_n \frac{\bar{H}(\epsilon_{n+1}) - \bar{H}(\epsilon_n)}{[\bar{H}(\epsilon_n)]^{1/p}} \\ &\leqslant C \sum_{n=N}^{\infty} \delta_n[\bar{H}(\epsilon_{n+1}) - \bar{H}(\epsilon_n)] \end{aligned} \qquad (2.6)$$

for some constant C, since $\bar{H}(\epsilon_n) > n$.

$$\begin{aligned}\bar{H}(\epsilon_{n+1}) - \bar{H}(\epsilon_n) &= \log \left[4 \prod_{k=M(n)+1}^{M(n+1)} \frac{b_k}{\delta_{n+1}} \left(\frac{\delta_n}{\delta_{n+1}}\right)^{M(n)}\right] \\ &= \log 4 + M(n) \log \frac{\delta_n}{\delta_{n+1}} + \sum_{k=M(n)+1}^{M(n+1)} \log \frac{b_k}{\delta_{n+1}}.\end{aligned}$$

Note that

$$\sum_{k=M(n)+1}^{M(n+1)} \log \frac{b_k}{\delta_{n+1}} \leqslant (M(n+1) - M(n)) \log \frac{b_{M(n)+1}}{\delta_{n+1}}$$
$$\leqslant (M(n+1) - M(n)) \log \frac{\delta_n}{\delta_{n+1}}.$$

Therefore

$$\bar{H}(\epsilon_{n+1}) - \bar{H}(\epsilon_n) \leqslant \log 4 + M(n+1) \log \frac{\delta_n}{\delta_{n+1}}$$
$$\leqslant \log 4 + M(n+1) \frac{\delta_n - \delta_{n+1}}{\delta_{n+1}},$$

and using (2.2) and (2.6),

$$\sum_{n=N}^{\infty} \delta_n[\bar{H}(\epsilon_{n+1}) - \bar{H}(\epsilon_n)] \leqslant C_1 \sum_{n=N}^{\infty} \delta_n + C_2 \sum_{n=N}^{\infty} n[\delta_n - \delta_{n+1}],$$

where C_1 and C_2 are constants. These series converge by Lemma 2.

To prove the converse, we assume that the integral is finite and show that the sum must be finite. This part of the proof is more difficult than the first part since we cannot form the sequence $\{b_k\}$ satisfying (2.2), and, consequently, cannot readily approximate the integral by sums.

Let $F(x) = a_k$, $k - 1 < x \leqslant k$ and $F^{-1}(y) = \sup\{x: F(x) \geqslant y\}$. For a fixed value of y, consider the partial sequence $\{a_k\}$, $k = 1,..., F^{-1}(y)$. Divide the interval $[-a_k, a_k]$ into $[2a_k/y + 1]$ disjoint intervals of length $y/2$. Then there are at least

$$\prod_{k=1}^{F^{-1}(y)} \frac{2a_k}{y}$$

disjoint sets, all contained in $B(\{a_k\})$, with radius $(F^{-1}(y))^{1/p}(y/4)$. Consequently,

$$H\left(B\{a_k\}, (F^{-1}(y))^{1/p} \frac{y}{4}\right) \geqslant \log 2^{F^{-1}(y)} \prod_{k=1}^{F^{-1}(y)} \frac{a_k}{y} \geqslant F^{-1}(y) \log 2. \quad (2.7)$$

Since $\{a_k\} \in l^p$, $\lim_{y \to 0} y(F^{-1}(y))^{1/p} = 0$. Let δ be such that $4\gamma = \delta F^{-1}(\delta) > yF^{-1}(y)$ if $0 \leqslant y < \delta$. The function $y(F^{-1}(y))^{1/p}$ is piecewise monotone. A sequence of continuous functions $\{G_k(y)\}$ can be obtained such that $y(G_k(y))^{1/p}$ is also piecewise monotone, and $y(G_k(y))^{1/p}$ increases to $y(F^{-1}(y))^{1/p}$ as $k \to \infty$. H, by definition, is nonincreasing; therefore, using (2.7),

$$H\left[\frac{y}{4}(G_k(y))^{1/p}\right] \geqslant H\left[\frac{y}{4}(F^{-1}(y))^{1/p}\right] \geqslant \log 2F^{-1}(y).$$

By Lemma 3,

$$\int_0^{\gamma} H^{1/q}(\epsilon)\, d\epsilon = \int_0^{\delta} H^{1/q}\left[\frac{y}{4}(G_k(y))^{1/p}\right] d\left[\frac{y}{4}(G_k(y))^{1/p}\right]$$

$$\geqslant C \int_0^{\delta} (F^{-1}(y))^{1/q}\, d[y(G_k(y))^{1/p}],$$

where $C = \log 2/4$. Integrating by parts, we obtain

$$\int_a^{\delta} [F^{-1}(y)]^{1/q}\, d[y(G_k(y))^{1/p}]$$

$$= (F^{-1}(\delta))^{1/q}\, \delta(G_k(\delta))^{1/p} - (F^{-1}(a))^{1/q}\, a(G_k(a))^{1/p}$$

$$- \int_a^{\delta} y(G_k(y))^{1/p}\, d(F^{-1}(y))^{1/q}$$

$$\geqslant -aF^{-1}(a) - \int_a^{\delta} y(G_k(y))^{1/p}\, d(F^{-1}(y))^{1/q}.$$

Assume that $\liminf_{a \to 0} aF^{-1}(a) \leqslant 4$, then

$$-\int_0^{\delta} y(G_k(y))^{1/p}\, d(F^{-1}(y))^{1/q} < C$$

independent of k. By the monotone convergence theorem and a change of variables,

$$\lim_{k \to \infty} - \int_0^{\delta} y(G_k(y))^{1/p}\, d(F^{-1}(y))^{1/q} = - \int_0^{\delta} y(F^{-1}(y))^{1/p}\, d(F^{-1}(y))^{1/q}$$

$$= \int_{F^{-1}(\delta)}^{\infty} F(x)\, x^{1/p}\, dx^{1/q}$$

$$= 1/q \int_{F^{-1}(\delta)}^{\infty} F(x)\, dx$$

$$= 1/q \sum_{F^{-1}(\delta)}^{\infty} a_k < C.$$

This proves the result when $\liminf_{a \to 0} aF^{-1}(a) \leqslant 4$, but this must be the case whenever $\int_0 H^{1/q}(\epsilon)\, d\epsilon < \infty$. To see this, suppose $\liminf_{a \to 0} aF^{-1}(a) \geqslant 4$; then for $k \geqslant N$ for some N sufficiently large, $a_k \geqslant 4/k$ and by (2.7),

$$H(k^{-1/q}) \geqslant H\left(\frac{k^{1/p} a_k}{4}\right) \geqslant \log 2k.$$

Therefore $\int_0 H^{1/q}(\epsilon)\, d\epsilon \geqslant C \sum k^{1/q}[k^{-1/q} - (k+1)^{-1/q}] = \infty$.

Proof of Theorem 2. Define $F(x) = b_k$, $k - 1 < x \leqslant k$; $F(0) = b_1$. Let $F^{-1}(y) = \sup\{x: F(x) \geqslant y\}$ and $E\{b_k\}$ be an ellipsoid in l^p. Mitjagin ([1], Theorem 3, p. 71) gives the following bounds for the ϵ-entropy of $E\{b_k\}$ with respect to the l^p metric:

$$\int_{2\epsilon}^{1} \frac{F^{-1}(y)}{y}\, dy \leqslant H(E\{b_k\}, \epsilon) \leqslant F^{-1}(\epsilon/2) \log 4 + \int_{\epsilon/2}^{1} \frac{F^{-1}(y)}{y}\, dy. \tag{2.8}$$

We also use a result which follows immediately from Lemma 1, namely that

$$\int_0 F^{-1}(cs^{1/p})\, ds < \infty \Leftrightarrow \int^{\infty} F^p(x)\, dx = \sum^{\infty} b_k{}^p < \infty, \tag{2.9}$$

where $c > 0$. Denote $H(E\{b_k\}, \epsilon)$ by $H(\epsilon)$. By (2.8),

$$\int_0^{\delta} H(\epsilon)\, d\epsilon^p \leqslant c_1 \int_0^{\delta} F^{-1}(\epsilon/2)\, d\epsilon^p + \int_0^{\delta} \int_{\epsilon/2}^{b_1} \frac{F^{-1}(y)}{y}\, dy\, d\epsilon^p$$

$$= c_1 \int_0^{\delta} F^{-1}(1/2s^{1/p})\, ds + \int_0^{\delta} \int_{s^{1/p}/2}^{b_1} \frac{F^{-1}(y)}{y}\, dy\, ds.$$

The first integral converges for $\{b_k\} \in l^p$ by (2.9). By Lemma 1, the second integral converges iff

$$\int_0^{\delta} sd \int_{s^{1/p}/2}^{b_1} \frac{F^{-1}(y)}{y}\, dy = -\frac{1}{p} \int_0^{\delta} F^{-1}\left(\frac{s^{1/p}}{2}\right) ds < \infty.$$

Consequently $\int_0^{\delta} H(\epsilon)\, d\epsilon^p < \infty$ if $\sum^{\infty} b_k{}^p < \infty$. Again by (2.6),

$$\int_0^{\delta} H(\epsilon)\, d\epsilon^p \geqslant \int_0^{\delta} \int_{2\epsilon}^{\delta_1} \frac{F^{-1}(y)}{y}\, dy\, d\epsilon^p = \int_0^{\delta} \int_{2s^{1/p}}^{b_1} \frac{F^{-1}(y)}{y}\, dy\, ds.$$

By Lemma 1, the last integral converges iff

$$\int_0^{\delta} sd \int_{2s^{1/p}}^{b_1} \frac{F^{-1}(y)}{y}\, dy = -\frac{1}{p} \int_0^{\delta} F^{-1}(2s^{1/p})\, ds < \infty.$$

Therefore if $\int_0^{\delta} H(\epsilon)\, d\epsilon^p < \infty$, then by (2.7), $\sum^{\infty} b_k{}^p < \infty$.

Proof of the Corollary. This result is contained in Corollary 4 of [6] but since the proof, in this case, is easy, it will be repeated. Define $F(x) = d_n$, $n - 1 < x \leqslant n$; $F(0) = d_1$; then

$$F^{-1}(y) = \sup\{x: F(x) \geqslant y\} = \sup\{n: d_n \geqslant y\} = M(y).$$

By Lemma 1,

$$\int_0 M^{1/q}(y)\,dy < \infty \Leftrightarrow \int_0 yd(F^{-1}(y))^{1/q} < \infty.$$

By a change of variables,

$$\int_0 yd(F^{-1}(y))^{1/q} = \int^{\infty} F(x)\,dx^{1/q} = 1/q \int^{\infty} \frac{F(x)}{x^{1/p}}\,dx$$

and

$$\int^{\infty} \frac{F(x)}{x^{1/p}}\,dx < \infty \Leftrightarrow \sum \left(\frac{1}{n}\sum_{k=n}^{\infty} a_k{}^p\right)^{1/p} < \infty \Leftrightarrow \{a_k\} \in l^1.$$

The last statement is Lemma 2.

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